Plasma Oscillations :

If the electron in a plasma are displaced from a uniform background of ions, electric field will be build up in such a direction as to restore the neutrality of the plasma by pulling the electrons back to their original positions. Because of their inertia, the electrons will overshoot and oscillate around their equilibrium positions with a characteristic frequency known as the plasma frequency. This oscillation is so fast that the massive ions do not have time to respond to the oscillating field and may be ~~deex~~ considered as fixed.

We shall derive an expression for the plasma frequency ω_p in the simplest case, making the following assumptions :

i) There is no magnetic field
ii) There are no thermal motions $(KT = 0)$
iii) The ions are fixed in space in a uniform distribution.
iv) The plasma is infinite in extent and
v) The electron motions occur only in the x-direction.

As a consequence of last assumption, we have

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} \qquad \vec{E} = \vec{E}\,\hat{x} \qquad \vec{\nabla}\times\vec{E} = 0 \qquad E = -\vec{\nabla}\phi \longrightarrow (1)$$

There is, therefore, no fluctuating magnetic field; this is an electrostatic oscillation.

The electron equations of motion and continuity are

$$m n_e \left[\frac{\partial \vec{v}_e}{\partial t} + (\vec{v}_e \cdot \vec{\nabla}) \vec{v}_e \right] = -e n_e \vec{E} \longrightarrow (2)$$

$$\frac{\partial n_e}{\partial t} + \vec{\nabla} \cdot (n_e \vec{v}_e) = 0 \longrightarrow (3)$$

The only Maxwell equation we shall need is the one that does not involve B: Poisson's equation. This is a high-frequency oscillation; electron inertia is important and the deviation from neutrality is the main effect in this perticular case. Consquently, we write

$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = \epsilon_0 \frac{\partial E}{\partial x} = e(n_i - n_e) \longrightarrow (4)$$

Now, we first separate the independent variables into two parts: an "equilibrium" part indicated by a subscript 0, and a "perturbation" part indicated by a subscript 1:

$$n_e = n_0 + n_1 \quad \vec{v}_e = \vec{v}_0 + \vec{v}_1 \quad \vec{E} = \vec{E}_0 + \vec{E}_1 \longrightarrow (5)$$

The equilibrium quantities express the state of the plasma in the absence of the oscillation. Since we have assumed a uniform neutral plasma at rest before the electrons are displaced, we have

$$\left. \begin{aligned} \vec{\nabla} n_0 = \vec{v}_0 = \vec{E}_0 = 0 \\ \frac{\partial n_0}{\partial t} = \frac{\partial \vec{v}_0}{\partial t} = \frac{\partial \vec{E}_0}{\partial t} = 0 \end{aligned} \right\} \longrightarrow (6)$$

Equation (2) becomes

$$m \left[\frac{\partial \vec{v}_1}{\partial t} + (\vec{v}_1 \cdot \overset{0}{\vec{\nabla}}) \vec{v}_1 \right] = -e \vec{E}_1 \longrightarrow (7)$$

The term $(\vec{v}_1 \cdot \vec{\nabla}) \vec{v}_1$ is seen to be quadratic in an amplitude quantity, and we shall linearize by neglecting it. The linear theory is valid as long as $|v_1|$ is small enough that such quadratic terms are indeed negligible.

similarly eqn. 3 becomes

$$\left.\begin{aligned} &\frac{\partial n_1}{\partial t} + \vec{\nabla}\cdot(n_0\vec{v}_1 + \cancel{n_1\vec{v}_1}^{\,0}) = 0 \\ &\frac{\partial n_1}{\partial t} + n_0\vec{\nabla}\cdot\vec{v}_1 + \vec{v}_1\cdot\cancel{\vec{\nabla} n_0}^{\,0} = 0 \end{aligned}\right\} \rightarrow (8)$$

In poisson's equation (4), we note that $n_{i0} = n_{e0}$ in equilibrium and that $n_{i1} = 0$ by the assumption of fixed ions, so we have

$$\epsilon_0 \vec{\nabla}\cdot\vec{E}_1 = -en_1 \longrightarrow (9)$$

The oscillating quantities are assumed to behave sinusoidally:

$$\left.\begin{aligned} \vec{v}_1 &= v_1 e^{i(kx-\omega t)}\hat{x} \\ n_1 &= n_1 e^{i(kx-\omega t)} \\ \vec{E} &= E e^{i(kx-\omega t)}\hat{x} \end{aligned}\right\} \longrightarrow (10)$$

The time derivative $\frac{\partial}{\partial t}$ can therefore be replaced by $-i\omega$ and the gradient $\vec{\nabla}$ by $ik\hat{x}$. Equation (7) to (9) now become

$$-im\omega v_1 = -eE_1 \longrightarrow (11)$$

$$-i\omega n_1 = -n_0 ikv_1 \longrightarrow (12)$$

$$ik\epsilon_0 E_1 = -en_1 \longrightarrow (13)$$

Eliminating n_1 and E_1, we have for equation (11)

$$-im\omega v_1 = -e\,\frac{-e}{ik\epsilon_0}\cdot\frac{-n_0ikv_1}{-i\omega} = -i\,\frac{n_0e^2}{\epsilon_0\omega}v_1 \rightarrow (14)$$

If v_1 does not vanish, we must have

$$\omega^2 = n_0e^2/m\epsilon_0$$

The plasma frequency is therefore

$$\boxed{\omega_p = \left(\frac{n_0e^2}{\epsilon_0 m}\right)^{1/2} \text{ rad/sec}} \longrightarrow (15)$$

Numerically, One can use the approximate formula

$$\frac{\omega_p}{2\pi} = f_p \approx 9\sqrt{n}$$

This frequency depending only on the plasma density, is one of the fundamental parameters of a plasma. Because of the smallness of m, the plasma frequency is usually very high. For instance, in a plasma of density $n = 10^{18}\ m^{-3}$, we have

$$f_p \approx 9(10^{18})^{1/2} = 9 \times 10^9\ sec^{-1} = 9\ GHz$$

Radiation of f_p normally lies in the microwave range. we can compare this with another electron frequency: ω_c. A usefull numerical formula is

$$f_{ce} \simeq 28\ GHz/Tesla$$

Thus if $B \approx 0.32\,T$ and $n \approx 10^{18}\ m^{-3}$, the cyclotron frequency is approximately equal to the plasma frequency for electrons.

Equation 15 tells us that if a plasma oscillation is to occur at all, it must have a frequency depending only on n. In particular, ω does not depend on K, so the group velocity $d\omega/dk$ is zero.